MODULE 2: ADDITION AND SUBTRACTION OF WHOLE NUMBERS

The likelihood that our shepherd of Module 1 had all his sheep grazing in one spot is rather small. More likely, the sheep were distributed in small clusters at various parts of the pasture. So in order to keep track of the total number of sheep, the shepherd would have to combine two or more smaller counts.

The process of combining these smaller counts to obtain a total count is known as the process of addition. Addition is the main topic of this module.

Example 1

Show how the shepherd would use tally marks to indicate the total number of sheep if five sheep were grazing in one part of the pasture and seven sheep were grazing in another part of the pasture.

Answer: ||||| |||||||

Using one tally mark for each sheep, the shepherd would write $\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}\ \underset{5}{|}$ to indicate the five sheep in one part of the pasture (This is the principle we used in Example 1 of Module 1).

The digits are supplied fo the sake of clarity. The shepherd wouldn't write the

Then he would write $\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}\ \underset{5}{|}\ \underset{6}{|}\ \underset{7}{|}$ to represent the seven sheep in the other part of the pasture.

He would then combine the two counts into one by writing:

$$\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}\ \underset{5}{|}\qquad \underset{6}{|}\ \underset{7}{|}\ \underset{8}{|}\ \underset{9}{|}\ \underset{10}{|}\ \underset{11}{|}\ \underset{12}{|}$$

The extra space between th fifth and sixth tally mark is to emphasize the two separate counts, but the extra spacing is not neces- ary. All that matters is the total number of tally marks.

Using modern numerals, we would say that the shepherd added 7 sheep to 5 sheep to get a total of 12 sheep.

To indicate that we added 7 to 5 and got 12 as the total we write: 5 + 7 = 12 and we say that we added 5 and 7.

Notice the order. We are starting with 5 and adding 7 to it. So we write 5 + 7. 7 + 5 would mean that we started with 7 and added 5 to it. We'll say more about this in the next example.

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Vocabulary

(1) "+" is called a plus sign. *It is read as "plus"*

(2) 5 + 7 is read as "5 plus 7" and is called the sum *of 5 and 7.*

(3) "=" is called an equal sign. *It is read as "equals" or "is".*

(4) So 5 + 7 = 12 is read as:

The sum of 5 and 7 is 12.
or
The sum of 5 and 7 equals 12.
or
Five plus seven is twelve.
or
Five plus seven equals twelve.

(5) In the sentence 5 + 7 = 12, 5 and 7 are called the terms *or* summands; *while 12 is called the* sum (of 5 and 7).

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Do not say "5 and 7"

In terms of grammar, think of "=" as meaning "is a synonym for". That is, when we write 5 + 7 = 12 it means that 5 + 7 and 12 are different numerals that name the same number (twelve).

It is not important that we were dealing with 5, 7, and 12. Rather the statement 5 + 7 = 12 tells is two much more important things. Namely:

(1) The sum of two whole numbers is always a whole number.

That is, 5 and 7 are two (different) whole numbers; but 5 + 7 is the single whole number, twelve (12).

(2) When we say 5 + 7 = 12 we are assuming that 5, 7, and 12 are adjectives modifying the *same* noun. For example 5 nickels plus 7 nickels is 12 nickels; 5 dimes plus 7 dimes is 12 dimes; but 5 nickels plus 7 dimes is neither 12 nickels or 12 dimes.

At least nickels and dimes have cents as a common denomination, but 5 miles plus 7 hours doesn't even make sense.

Example 2

What is the sum of 7 and 5?

Answer: 12

Since the 7 is written first, we may think in terms of $\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}\ \underset{5}{|}\ \underset{6}{|}\ \underset{7}{|}\quad \underset{8}{|}\ \underset{9}{|}\ \underset{10}{|}\ \underset{11}{|}\ \underset{12}{|}$

In terms of "fill-in-the-blank" the question is:
7 + 5 = ____

So the sum of 7 and 5 is 12. We might also say that 7 plus 5 is (equals) 12.

That is, 7 + 5 = 12

Comparing Examples 1 and 2 we notice that while 5 + 7 and 7 + 5 represent a change in order, the sum is the same. In terms of tally marks, whether we write

| | | | | | | | | | | |

or

| | | | | | | | | | | |

the total number of tally marks is the same.

This is a more significant result than at first may mee the eye. In most real-life situations, the result does depend on order. For example it is easier to take 2 coins from a collection o 6 than it is to take 6 coin from a collection of 2!

So when we write 5 + 7 it makes no difference in the answer (sum) whether we view it as adding 5 to 7 or adding 7 to 5.

This can be a very helpful piece of information.

Example 3

Find the sum of 39 and 3.

Answer: 42

We could write 39 tally marks followed by 3 more tally marks. An easier way is to count, starting with 39. That is, as shown below, write 39, followed by 3 "blanks"

39 __ __ __

Then count in sequence to obtain:

39 <u>40</u> <u>41</u> <u>42</u>

Using blanks is like counting on your fingers. In fact if we use tally marks rather than blanks to stand for fingers we have:

39 $\underset{40}{|}\ \underset{41}{|}\ \underset{42}{|}$

Do you see how much more cumbersome it would be if we had been asked to add 39 to 3? It is indeed

helpful that we may view 3 + 39 as being equivalent to 39 + 3.

Example 4

Fill in the blank:

$4 + 57 =$ ____.

Answer: 61

Rather than start with 4 and add 57 blanks, we can start with 57 and add 4 blanks.

Youngsters often count on their fingers, using the numbers in the order they appear. For this reason, it is easier for them to do 57 + 4 than 4 + 57--and they are not born with the knowledge that 57 + 4 and 4 + 57 are equal.

Proceeding as we did in Example 3, we get:

57 58 59 60 61

The fact that we may replace 4 + 57 by 57 + 4 without changing the answer is important enough to be given a special name in mathematics.

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Vocabulary

If f and s stand for any two numbers, then $f + s = s + f$. The fact that we can change the order of the summands without changing the sum is known as

the commutative property
of
addition

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Think of f as standing for the first number and s for the second number. This generalizes our above discussion beyond the specific pairs of numbers 4 and 57, 5 and 7, or 39 and 3.

Notice that if the number of blanks is not in excess of our number of fingers, we often count on our fingers to get the answer. Perhaps this is the way you first learned your addition tables. By way of review:

This is another reason why 0,1,2,3,4,5,6,7,8, and 9 are called digits. We add them using our fingers (digits).

Example 5

How much is 6 + 9?

Answer: 15

This is another way of being asked to find the sum of 6 and 9. If we wanted to use the tally mark method of Example 1, we could write:

| | | | | | | | | | | | | | |
1 2 3 4 5 6   7 8 9 10 11 12 13 14 15

We could also write:

$6 \quad \underset{1}{\underline{7}} \quad \underset{2}{\underline{8}} \quad \underset{3}{\underline{9}} \quad \underset{4}{\underline{10}} \quad \underset{5}{\underline{11}} \quad \underset{6}{\underline{12}} \quad \underset{7}{\underline{13}} \quad \underset{8}{\underline{14}} \quad \underset{9}{\underline{15}}$

A quicker way would be to use the commutative property of addition and write:

That is, 6 + 9 = 9 + 6

$9 \quad \underset{1}{\underline{10}} \quad \underset{2}{\underline{11}} \quad \underset{3}{\underline{12}} \quad \underset{4}{\underline{13}} \quad \underset{5}{\underline{14}} \quad \underset{6}{\underline{15}}$

Or, using our "fingers":

9 | | | | | |
10 11 12 13 14 15

The key point is that we can do Example 5 by counting on our fingers. In a similar manner we can construct a table showing us how to find the sum of any two single-digit numbers. The table is shown as Figure 1.

second number (across the top); *first number* (down the side)

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

(Figure 1)

How to read the table:

Suppose we want to find how much 5 + 3 is.

(1) Look down the first column and locate the row headed by "5"

(2) Look along the first row and locate the column headed by "3"

(3) The answer is the number that is common to both row and column:

+	0	1	2	3	4
0					
1					
2					
3					
4					
5				8	

You may wonder why we bother with 0 in the table; but as we have already seen, 0 plays a very important role in place value. In fact, the role of 0 is so important that we give "adding 0" a special name:

> *Vocabulary*
>
> *0 is called the* additive identity.
>
> *Namely, adding 0 to a number does not change the "identity" of that number.*
>
> *More symbolically, if n denotes any number, then:*
>
> $$n + 0 = n$$

Example 6

What is the sum of 673 and 0?

Answer: 673

If we add nothing to 673 then we still have 673. In terms of the shepherd idea, if there are 673 sheep in one part of the pasture and none anywhere else; then there are 673 sheep in all.

Ordinarily we wouldn't start counting until we had at least one; but with place value we have to use 0 as a digit.

The important thing about place value is that once we know the table indicated in Figure 1, *we can add any whole numbers*. The point is, for example, that once we know that 6 + 9 = 15, we also know that the sum of 6 hundred and 9 hundred is fifteen hundred; that the sum of 6 thousand and 9 thousand is 15 thousand, and so on. As we shall see beginning with the next example, this is very helpful.

That is, when we say that 6 + 9 = 15, we are assuming that 6, 9, and 15 are all adjectives that modify the same noun

Example 7

What is the sum of 2,000,000,000 and 3,000,000,000?

Answer: 5,000,000,000

2,000,000,000 means 2 *billion*

and

3,000,000,000 means 3 *billion*

Since 2 + 3 = 5, it follows that

2 *billion* + 3 *billion* = 5 *billion*; or,

in place value notation:

2,000,000,000 + 3,000,000,000 = 5,000,000,000

To allow place value to keep track of the denominations for us, we often use the vertical notation:

$$\begin{array}{r} 2,000,000,000 \\ +\ \underline{3,000,000,000} \\ 5,000,000,000 \end{array}$$

Horizontal notation has become more popular since the advent of calculators. That is, you don't have to "line up" the powers of ten on the calculator; the calculator does it by itself.

Example 8

What is the sum of 231 and 456?

Answer: 687

It would be brutal to have to write 231 tally marks, followed by an additional 456 tally marks.

In terms of Roman numerals, we're adding CCXXXI and CCCCXXXXXIIIIII. Combining we get: CCCCXXXXXIIIIII
CCXXXI
or:
CCCCCCXXXXXXXXIIIIIII

Instead we write the nouns the digits modify. Namely:

	hundreds	tens	ones
	2	3	1
(+)	4	5	6
	6	8	7

Get the idea? 2 hundred + 4 hundred is 6 hundred; 3 tens + 5 tens is 8 tens; and 1 one + 6 ones is 7 ones.

Using the nouns properly, we got the answer simply by knowing: 2 + 4 = 6, 3 + 5 = 8, and 1 + 6 = 7

Using the vertical form, we'd simply write:

$$\begin{array}{r} 231 \\ +\ \underline{456} \\ 687 \end{array}$$

But don't forget: The adjectives have to modify the same noun.

Example 9

Find the sum of 2 thousand and 3 hundred.

Answer: 2,300

Using the powers of ten we have:

	thousands	hundreds	tens	ones
	2	0	0	0
(+)	0	3	0	0
	2	3	0	0

The 0's are included for emphasis. Notice how we use the facts that 2 + 0 = 2; 3 + 0 = 0; and 0 + 0 = 0.

In this example we had:

2 *thousands* + 3 *hundreds*.

See?
2 thousand + 3 thousand = 5 thousand

2 hundred + 3 hundred = 5 hundred
but
2 thousand + 3 hundred is neither 5 hundreds nor 5 thousands.

The answer wasn't "5" because 2 and 3 were modifying different nouns.

The point is that by working with one power of ten at a time, addition is reduced to consecutive sums of single digits. The problem is that in place value notation we cannot have more than nine of any denomination (power of ten).

Example 10

Find the sum of 57 and 68.

Answer: 125

We proceed as before to obtain:

hundreds	tens	ones
	5	7
	6	8
	11	15

See the problem? If we just wrote the digits in the answer without the nouns, it would look like 1115 and this would be confused with 1,115 (instead of being read as 11 tens and 15 ones)

However we can exchange 10 ones for 1 ten and 10 tens for 1 hundred to get:

hundreds	tens	ones
	5	7
	6	8
	11	15
	12	5
1	2	5

While each of the last three rows name the same number, only the last row is permitted when we use place value notation.

Rather than do all this writing, we have an abbreviated way for representing the vertical form. Namely, we write the problem as before:

$$\begin{array}{r} 57 \\ +\ \underline{68} \end{array}$$

Looking at the ones place, we say:

> "8 plus 7 is 15. 'Bring down' the 5, and 'carry' the 1." *As we say this we write the 5 beneath the 8, and the 1 above the 5 in the tens-place. Thus:*

$$\begin{array}{r} 1 \\ 57 \\ +\ \underline{68} \\ 5 \end{array}$$

We can write 8 + 7 = 15 because both 8 and 7 modify the ones-place.

The phrase "Bring down the 5 and carry the 1" merely says to keep 5 of the 15 ones and exchange the other 10 ones for 1 ten.

We then move to the tens-place and add the 1, 5, and 6 to get:

$$\begin{array}{r} 1 \\ +\ \underline{5} \\ 6 \\ +\ \underline{6} \\ 12 \end{array}$$

Again we can add the 1, 5, and 6 becaue all three digits modify the tens-place

We write the 12 in the tens-place by placing the 2 underneath the 6. That is:

$$\begin{array}{r} 1 \\ 57 \\ +\ \underline{68} \\ 125 \end{array}$$

Once you understand that in place value we can only have a single digit per power of ten, it should be easy to see that this method is just an abbreviation for what we did in Example 10.

But this approach has introduced a new problem. For the first time in this module we added more than two terms. Recall that our addition table only showed how we could add two (single-digit) terms.

The next example will indicate how we can extend the idea of adding two terms . Namely we shall see that no matter how many terms there are, we need only add two at a time.

Example 11

A shepherd has three sheep in one part of the pasture, four sheep in another part of the pasture, and five sheep in a third part of the pasture. Show how the shepherd could indicate the total number of sheep by using tally marks.

Answer: ||| |||| |||||

In essence, this is much like Example 1.

Using one tally mark for each sheep, he would write $\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}$ to indicate the three sheep in one part of the pasture; he'd write $\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}$ to indicate the four sheep in the second part of the pasture; and he'd write $\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}\ \underset{5}{|}$ to indicate the five sheep in the third part of the pasture.

Again, the digits are supplied only for clarity and would not have been used by the shepherd.

Combining the tally marks, he'd have

$$\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\qquad \underset{4}{|}\ \underset{5}{|}\ \underset{6}{|}\ \underset{7}{|}\qquad \underset{8}{|}\ \underset{9}{|}\ \underset{10}{|}\ \underset{11}{|}\ \underset{12}{|}$$

If we wanted to write Example 11 in the modern language of place value, we might write:

$$3 + 4 + 5 = 15$$

But again notice that the shepherd could have counted the groups in any order. Even in the given order, he could have used the following two different groupings:

$$\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\ \underset{4}{|}\ \underset{5}{|}\ \underset{6}{|}\ \underset{7}{|}\qquad \underset{8}{|}\ \underset{9}{|}\ \underset{10}{|}\ \underset{11}{|}\ \underset{12}{|}\qquad (1)$$

or

$$\underset{1}{|}\ \underset{2}{|}\ \underset{3}{|}\qquad \underset{4}{|}\ \underset{5}{|}\ \underset{6}{|}\ \underset{7}{|}\ \underset{8}{|}\ \underset{9}{|}\ \underset{10}{|}\ \underset{11}{|}\ \underset{12}{|}\qquad (2)$$

In mathematics we use parentheses in much the same way that hyphens are used in English. In (1) we first added 3 and 4 and then added 5. We'd write this as:

$$(3 + 4) + 5 \qquad (3)$$

In (2) we first added 4 and 5 and then added this to 3. We'd write this as:

$$3 + (4 + 5) \qquad (4)$$

The same kind of problem exists in language. Look at the phrase:
the high school building
We can interpret it as:
the high-school building
or as:
the high school-building
The two meanings are quite different, even though the words are in exactly the same order. If we used parentheses instead of hyphens, we'd write either the (high school) building or the high (school building)

Although the groupings in (3) and (4) are different, both represent the same number. Namely;

$$\left.\begin{array}{l} (3 + 4) + 5 = \\ \quad\downarrow \\ \quad 7 \quad + 5 = \end{array}\right\} \qquad \left\{\begin{array}{l} 3 + (4 + 5) = \\ \qquad\quad\downarrow \\ 3 + \quad 9 \quad = \end{array}\right.$$

$$12 \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow 12$$

The fact that the sum of three numbers does not depend on how we group the three numbers is an important enough fact to warrent a special name:

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Definition

For any three numbers f, s, and t $f + s + t$ is the same whether we read it as $(f + s) + t$ or as $f + (s + t)$. That is:

$$f + (s + t) = (f + s) + t.$$

this is known as:

the associative property
of
addition

****

While we're stating this in terms of three numbers, you should also notice that for any number of terms, the sum does not depend on the grouping of the terms.

Note:

Don't confuse the commutative property of addition with the associative property of addition. The commutative property allows us to change the order of the numbers; while the associate property allows us to change the grouping (but the order stays the same).

For example, the fact that $(3 + 4) + 5 = (4 + 3) + 5$ is the commutative property. If the parentheses are removed we can see that the order of the numbers is different in the two sums.

But when we write: $(3 + 4) + 5 = 3 + (4 + 5)$ we are using the associative property. Namely when the parentheses are removed, the order of the numbers is the same in each sum.

Example 12

How much is 4 + 7 + 3?

We can read this problem as if it were:

$$(4 + 7) + 3,$$

in which case we get:

$$(4 + 7) + 3 = 11 + 3$$
$$= 14$$

Everything within parentheses is one number. Hence 4 + 7 can be replaced by 11.

We could also have read the problem as:

$$4 + (7 + 3),$$

Notice that in this approach we never add more than two numbers at a time.

in which case we'd get:

$$4 + (7 + 3) = 4 + 10$$
$$= 14$$

We could use the commutative property here and replace 4 + 10 by 10 + 4.

The same idea works with for any number of digits.

Example 13

How much is 23 + 14 + 42?

Answer: 79

We can first add 23 and 14 and then add 42. That is, we can read the problem as:

$$\begin{array}{l}(23 + 14) + 42 = \\ \quad\downarrow \\ \quad 37 \quad\; + 42 = \\ \\ \qquad 79\end{array}$$

See how we add two terms at a time? We already know that:

$$\begin{array}{r}23\\+14\\\hline 37\end{array} \text{ and that } \begin{array}{r}37\\+42\\\hline 79\end{array}$$

We could also have read the problem as:

$$\begin{array}{l}23 + (14 + 42) = \\ \qquad\qquad \downarrow \\ 23 + \qquad 56 \qquad = \\ \\ \qquad 79\end{array}$$

$$\begin{array}{r}14\\+\ 42\\\hline 56\end{array} \text{ and } \begin{array}{r}23\\+\ 56\\\hline 79\end{array}$$

Let's now add two terms that have several digits and we'll leave more complicated problems to the Self-Test in the Study Guide.

Example 14

Find the sum of 34,567 and 27,298.

Answer: 61,865

In terms of powers of 10, we have:

	10,000	1,000	100	10	1
	3	4	5	6	7
(+)	2	7	2	9	8
	5	11	7	15	15

See how we're never adding more than two single-digits at a time?

This tells us that the sum is 5 ten-thousands, 11 thousands, 7 hundreds, 15 tens and 15 ones. The problem is that when we omit the nouns (denominations) we can only have single-digit adjectives. So we take the above answer and keep exchanging 10 of any power of ten for 1 of the next greater power of ten. That is:

It would be confusing, to say the least, if we left the answer as 51171515, although we might write it as

(5)(11)(7)(15)(15)

using the parenthese to hold the place-value.

	10,000	1,000	100	10	1
	3	4	5	6	7
(+)	2	7	2	9	8
	5	11	7	15	15
	5	11	7	16	5
	5	11	8	6	5
	6	1	8	6	5

15 ones = 1 ten + 5 ones
16 tens = 1 hundred + 6 tens
eleven 100s = one 1000 + one 10

In vertical form we write the problem → → → →

$$\begin{array}{r} 34{,}567 \\ +\ \underline{27{,}298} \end{array}$$

(1) Starting in the ones-place we add 7 and 8 to get 15 ones. We keep (bring down) the 5 and add 1 to the tens-place → → → → → → → → → → → → →

$$\begin{array}{r} {\scriptstyle 1} \\ 34{,}567 \\ +\ \underline{27{,}298} \\ 5 \end{array}$$

We say "7 plus 8 is 15; bring down the 5 and carry the 1"

(2) Now we go to the tens-place and add 1, 6, and 9 . That is:
$1+6+9=(1+6)+9=7+9=16$.
We bring down the 6 and carry 1 → → → → →

$$\begin{array}{r} {\scriptstyle 11} \\ 34{,}567 \\ +\ \underline{27.298} \\ 65 \end{array}$$

We're adding 1 ten, 6 tens and 9 tens to get 16 tens

(3) Now we move to the hundreds-place and add 1, 5, and 2 to get 8.
$[1+5+2=(1+5)+2=6+2=8]$ → → → → → → →

$$\begin{array}{r} {\scriptstyle 11} \\ 34{,}567 \\ +\ \underline{27{,}298} \\ 865 \end{array}$$

Since we only have 8 in the hundreds-place, we don't carry.

(4) Moving to the thousands-place we now add 4 and 7 to get 11. We bring down a 1 and carry a 1 to get→ → → → → → → → → → →

$$\begin{array}{r} {\scriptstyle 1\ \ 11} \\ 34{,}567 \\ +\ \underline{27{,}298} \\ 1{,}865 \end{array}$$

That is, we exchanged ten 1,000s for one 10,000.

(5) Moving to the ten-thousands-place, we add 1, 3, and 2 to get 6. → → → → → →

$$\begin{array}{r} {\scriptstyle 1\ \ 11} \\ 34{,}567 \\ +\ \underline{27{,}298} \\ 61{,}865 \end{array}$$

The study of addition brings with it the study of *subtraction*. Have you ever made change for anyone? Or have you ever watched anyone make change for you?

Suppose you go to a store and buy something for \$27. You get permission to pay for it by using a check for \$45. To give you your change the clerk subtracts \$27 from \$45. *Yet, unless the clerk has has an electronic cash register that subtracts, the clerk computes your change by using* addition.

For example, the clerk starts by saying "\$27". Then he takes three \$1-bills and says "and 3 is 30". Then he takes a \$10-bill and says "and 10 more is 40". Then he takes a \$5-bill and says "and 5 more is 45"

All in all he has given you three \$1-bills, one \$10-bill and a \$5-bill; for a total of \$18.

The notion of "subtraction" or "take away" was probably induced by a tally mark interpretation.

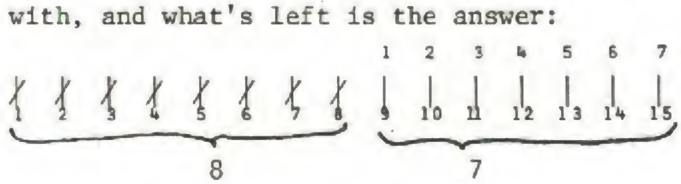
Start with.........\$27
Add................\$ 3
to get.............\$30
Add................\$10
to get.............\$40
Add................\$ 5
to get.............\$45

Example 15

What number must we add to 8 in order to get 15 as the sum?

We could start with the sum and represent it by fifteen tally marks.

Then cross out the eight you started with, and what's left is the answer:



Answer: 7

Read the problem carefully. You are not being asked to add 8 and 15! In terms of fill-in-the-blank, don't confuse:

(a) 8 + 15 = ___

with

(b) 8 + __ = 15

(a) is an addition problem; (b) is a subtraction problem

We could also have used our addition table. The key point is that (b) is true when the blank is replaced by 7. That is, 8 + 7 = 15

Using Example 14 as our backdrop, we can now introduce the following new vocabulary.

Notation and Vocabulary
for
Subtraction

(1) *When we want the number that must be added to 8 to yield 15 as the sum we write 15 - 8.*

(2) *" -" is called a* minus sign *and we read 15 - 8 as "15 minus 8". We also say that we're* subtracting *8 from 15.*

(3) *Since we know that 7 must be added to 8 to get 15 as the sum, we write:*

$$15 - 8 = 7$$

(4) *The answer to a subtraction problem is called the* difference. *That is, the difference of 15 and 8 is 7.*

Special Note:

Beware of order when you subtract. In terms of whole numbers we can take 8 from 15, but (at least as of now) we can't take 15 from 8. *That is, subtraction* doesn't *possess the* commutative property. *Remember:*

(5) *first number - second number means the number we must* add *to the* second *number to get the* first *number as the sum.*

15 and 8 are also given names. Using vertical form, we have:

$$\begin{array}{rl} 15 & \textit{(minuend)} \\ -\ 8 & \textit{(subtrahend)} \\ \hline 7 & \textit{(difference)} \end{array}$$

Nor does subtraction possess the associative property. For example:

$9 - (3 - 1) = 9 - 2 = 7$

but

$(9 - 3) - 1 = 6 - 1 = 5$

In other words, 9 - (3 - 1) and (9 - 3) - 1 look alike if the parentheses are left out but they name different numbers. Therefore, when we subtract, grouping is very important.

Again we need only know the addition tables for single-digit numerals in order to subtract if we use place value notation.

Let's practice with a few problems.

Example 16

What number must be added to 432 to give 758 as the sum?

Answer: 326

In terms of fill-in-the-blank, we have:

$$\_\_\_\_ + 432 = 758 \qquad (1)$$

and in terms of subtraction we have

$$758 - 432 = \_\_\_\_\_\_ \qquad (2)$$

In terms of a calculator, the advantage of (2) over (1) is that we can do (2) on the calculator. We enter 758, press the "-" key, enter 432, and then press the "=" key; but the calculator doesn't help us with (1).

If we write the problem in vertical form, with the powers of ten present, we have:

	hundreds	tens	ones
	7	5	8
−	4	3	2
	3	2	6

7 - 4 = 3 because 3 + 4 = 7
5 - 3 = 2 because 2 + 3 = 5
8 - 2 = 6 because 6 + 2 = 8

With the powers of ten omitted, we have:

$$\begin{array}{r} 758 \\ -\ \underline{432} \\ 326 \end{array}$$

As a check we need only add 326 to 432 and verify that the sum is 758.

That is: $\begin{array}{r} 758 \\ \underline{432} \\ 326 \end{array} \begin{array}{l} \}= \\ \}+ \end{array}$

Notice that we could also have used the change-making technique had we so desired.

Start with 432
Add...................300 → 300
to get................732
Add................... 20 → + 20
to get................752
Add.................... 6 → + 6
to get................758   326

There are many different ways that we can "make change" What's important is that we're adding on to 432 the amount necessary to give us 758 as the sum.

While the "vertical" method is more compact than the "change-making" method, the latter method allows us to visualize better that subtraction is really a form of addition.

There is one problem that occurs with subtraction when we deal with whole numbers. The least whole number is 0. So, for example, no matter what whole number we use to replace the blank in 9 + __, the sum has to be at least 9. In particular, then, it is impossible to fill in the blank with a correct whole number if we have 9 + ___ = 4.

That is, the least we could add is nothing, and that would still give us 9 as th answer. If we added any other whole number, the sum would have to be more than 9

In the language of subtraction, we cannot use whole numbers to solve the problem:

$$4 - 9 = \_\_\_\_$$

We can't take away more tha we have!

and this can cause us difficulty when we use the vertical method to subtract.

<u>Example 17</u>

Subtract 19 from 34.

Answer: 15

This is another was of asking us to compute 34 - 19. This, in turn, asks to find the number we must add to 19 to get 34 as the sum. There are several permissible methods, among which are:

<u>Tally Marks</u>

Start with 34 tally marks; cross-out 19 of them; and what's left is the answer:

XXXXXXXXXXXXXXXXXXX|||||||||||||||

Can you see how cumbersome this method can become? It's easy to visualize but its hardly feasible when we deal with larger numbers.

<u>Change-making</u>

Start with................19
Add.......................<u>5</u> ←
to get....................24
Add.......................<u>10</u> ←
to get....................34

5 + 10 = <u>15</u>

But suppose we wanted to use the vertical

method. We'd write:

	tens	ones
	3	4
−	1	9

The problem is that we can't subtract 9 ones from 4 ones! We can, however, exchange 1 ten for 10 ones. In more rigorous fashion:

$$
\begin{aligned}
34 &= 3 \text{ tens} + 4 \text{ ones} \\
&= 30 + 4 \\
&= (20 + 10) + 4 \\
&= 20 + (10 + 4) \\
&= 20 + 14 \\
&= 2 \text{ tens} + 14 \text{ ones.}
\end{aligned}
$$

We replaced 30 by 20 + 10

We used the Associative Property of Addition

In other words, we rewrite

tens	ones
3	4

as

tens	ones
2	14

Think in terms of money. You have three $10-bills and four $1-bills. You exchange a $10-bill for ten $1-bills. You now have two $10-bills and fourteen $1-bills.

Now the problem becomes:

	tens	ones
	2	14
−	1	9
	1	5

If the labels are omitted we have to find a way to make it clear that we have 14 ones. This is a touchy problem because we've already emphasized that we can't have more than a single-digit per place value column. What we do is write the 1 very small and a bit above the 4 to create the impression that there is still only one digit in the column. That is

$2\ {}^{1}4$

We might also have written 2(14) but that might be even more confusing.

Rather than write 34 and then replace it by $2\,^{1}4$, we use an abbreviation. We cross out the 3 and write the 2 above it, then we place the "small" 1 above and to the left of 4. That is:

$$\begin{array}{c} 2 \\ \cancel{3}\,^{1}4 \end{array}$$

When we cross out the 3 and replace it by 2, we say we're *borrowing* one from the three. In actuality, "borrowing" means that we've exchanged 1 of any power of ten for 10 of the next lower power of ten.

This is the "opposite" of carrying. When we carried, we exchanged 10 of a power of ten for 1 of the next greater power of ten.

In terms of borrowing, Example 16 would look like:

$$\begin{array}{rcc} & 2 & \\ & \cancel{3} & ^{1}4 \\ - & 1 & 9 \\ \hline & 1 & 5 \end{array}$$

Now we're subtracting 9 from 14 and 1 from 2.

Let's practice a bit more with the borrowing technique.

Example 18

How much is 543 - 187?

Answer: 356

If we align the numbers vertically we get:

$$\begin{array}{rccc} & 5 & 4 & 3 \\ - & 1 & 8 & 7 \\ \hline \end{array}$$

We cannot subtract 7 from 3, so we go to the tens-place and borrow 1 from the 4. →

$$\begin{array}{rccc} & & 3 & \\ & 5 & \cancel{4} & ^{1}3 \\ - & 1 & 8 & 7 \\ \hline \end{array}$$

We now return to the ones-place and subtract 7 from 13 to get → → → → → → → → → → → → → →

$$\begin{array}{rccc} & & 3 & \\ & 5 & \cancel{4} & ^{1}3 \\ - & 1 & 8 & 7 \\ \hline & & & 6 \end{array}$$

Next we repeat the procedure, starting with the tens-place. We can't subtract 8 from 3, so we go to the hundreds-place and borrow 1 from the 5 to get → → → → → → → → → → → → → → →

$$\begin{array}{rrr} & 4 & {}^{1}3 & \\ & \cancel{5} & \cancel{4} & {}^{1}3 \\ - & 1 & 8 & 7 \\ \hline \end{array}$$

We then subtract 8 tens from 13 tens to get →

$$\begin{array}{rrrr} & 4 & {}^{1}3 & \\ & \cancel{5} & \cancel{4} & {}^{1}3 \\ - & 1 & 8 & 7 \\ \hline & & 5 & 6 \end{array}$$

Finally, we go to the hundreds-place and subtract 1 from 4 to get → → → → → → → → → → → → → →

$$\begin{array}{rrrr} & 4 & {}^{1}3 & \\ & \cancel{5} & \cancel{4} & {}^{1}3 \\ - & 1 & 8 & 7 \\ \hline & 3 & 5 & 6 \end{array}$$

The previous steps are included for the sake of illustration. If we were doing the problem ourselves, all we'd actually see on the paper is the last step.

If the place-values were present, we'd see that what we've really done by borrowing is to rewrite 543 as 400 + 130 + 13. We then had:

	400	130	13
(-)	100	80	7
	300	50	6

That is:

hundreds	*tens*	*ones*	
5	*4*	*3*	=
5	*3*	*13*	=
4	*13*	*13*	
- 1	*8*	*7*	
3	*5*	*6*	

Note that borrowing is simply a technique used in place value. We can subtract without borrowing. In terms of change-making we have:

Start with.............187
Add....................... 6 ←
to get. 193
Add..................... 50 ←
to get..................243
Add....................300 ←
to get..................543

$$\begin{array}{rr} & 6 \\ + & 50 \\ + & 300 \\ \hline & 356 \end{array}$$

No matter how many digits we have, the procedure remains the same. We do the same sequence of steps repeatedly. The borrowing method does bother some people when we have to borrow from a place in which there is a 0. That is, we can't borrow if there's nothing to borrow. Yet if we understand what we're doing, this doesn't present a major problem.

Example 19

Use borrowing to subtract 2,678 from 7,305.

Answer: 4,627

Using the vertical alignment we have:

$$\begin{array}{r} 7\ 3\ 0\ 5 \\ -\ \underline{2\ 6\ 7\ 8} \end{array}$$

We can't subtract 8 from 5. So we go to the tens-place to borrow. But the 0 there tells us that there are no tens to borrow. So we go to the hundreds-place. We "borrow" one of our 3 hundreds, exchanging it for 10 tens. (See margin diagrams) → → → → → → → → → → → → → →

$$\begin{array}{r} 2 \\ 7\ \cancel{3}\,{}^{1}0\ 5 \\ -\ \underline{2\ 6\ 7\ 8} \end{array}$$

Now we can borrow one of the 10 tens and exchange it for 10 ones. → → → → → → → → → → →

$$\begin{array}{r} 2\ 9 \\ 7\ \cancel{3}\,{}^{1}0\,{}^{1}5 \\ -\ \underline{2\ 6\ 7\ 8} \end{array}$$

Returning to the ones-place we can now subtract 8 from 15 to obtain → → → → → → → → → → →

$$\begin{array}{r} 2\ 9 \\ 7\ \cancel{3}\,{}^{1}\cancel{0}\,{}^{1}5 \\ -\ \underline{2\ 6\ 7\ 8} \\ 7 \end{array}$$

Next we move to the tens-place and subtract 7 from 9 to get → → → → → → → → → → → → →

$$\begin{array}{r} 2\ 9 \\ 7\ \cancel{3}\,{}^{1}\cancel{0}\,{}^{1}5 \\ -\ \underline{2\ 6\ 7\ 8} \\ 2\ 7 \end{array}$$

Moving to the hundreds place, we can't subtract 6 from 2, so we go to the thousands-place and borrow one from the 7 to get → → → → → →

$$\begin{array}{r} 6\,{}^{1}2\ 9 \\ \cancel{7}\ \cancel{3}\,{}^{1}\cancel{0}\,{}^{1}5 \\ -\ \underline{2\ 6\ 7\ 8} \\ 2\ 7 \end{array}$$

(Notice that the 1 went next to the 2, not the 3 Once we cross ou a digit, it's gon

Coming back to the hundreds-place, we now subtract 6 from 12 to get → → → → → → → → → → → → →

$$\begin{array}{r} 6\,{}^{1}2\ 9 \\ \cancel{7}\ \cancel{3}\,{}^{1}\cancel{0}\,{}^{1}5 \\ -\ \underline{2\ 6\ 7\ 8} \\ 6\ 2\ 7 \end{array}$$

Finally, we go to the thousands place and subtract 2 from 7 to get → → → → → → → → → → →

$$\begin{array}{r} 6\,{}^{1}2\ 9 \\ \cancel{7}\ \cancel{3}\,{}^{1}\cancel{0}\,{}^{1}5 \\ -\ \underline{2\ 6\ 7\ 8} \\ 4\ 6\ 2\ 7 \end{array}$$

The final step shows us, in an abbreviated form, that we rewrote 7,305 as 6 thousands, 12 hundreds, 9 tens, and 15 ones. That is:

$$\begin{array}{rrrrrrrr} & 6000 & + & 1200 & + & 90 & + & 15 \\ (-) & \underline{2000} & & \underline{600} & & \underline{70} & & \underline{8} \\ & 4000 & + & 600 & + & 20 & + & 7 \end{array}$$

While the method used in these examples is efficient it is also frustrating. The problem is that we get the least important digits first, and this problem worsens as the number of digits increases.

Suppose, for example, that we used the same method to find the sum of 2,119,435,982 and 4,967,059,231. Our first step would yield:

$$\begin{array}{r} 2{,}119{,}435{,}982 \\ +\ \underline{4{,}967{,}059{,}231} \\ 3 \end{array}$$

and not until *nine steps later* do we arrive at:

$$\begin{array}{r} 1\ \ \ 1\ \ 11\ \ 1\ \ \ \ \ \ \ \ \\ 2{,}119{,}435{,}982 \\ +\ \underline{4{,}967{,}059{,}231} \\ 7{,}086{,}495{,}213 \end{array}$$

Think of $7,086,495,213. If the 3 is in error, we're off by at most a few dollars. But if the 7 is in error, we are off by billions *of dollars.*

It would be nice if we could develop the *number* sense to expect the sum to be approximately 7 billion before we even began to find the exact answer.

A very important type of approximation is known as *rounding off*. To understand rounding off, it is helpful to develop some new vocabulary.

- - *The Multiples of Ten*

In the counting sequence 1, 2, 3, . . . think of the numbers as modifying *ten*.

In mathematics we often use ". . ." as an abbreviation for "and so on"

The resulting numbers:

1 *ten*, 2 *tens*, 3 *tens*, 4 *tens*, . . .

are called *the multiples of ten*.

10 is called the 1st multiple of ten, 20 is called the 2nd multiple of 10, 30 is called the 3rd multiple of ten and so on.

Get the idea? For example 30 is 3 tens. So we call 30 the 3rd multiple of ten.

Let's make sure we understand the new notation.

Example 20

What is the 23rd multiple of ten?

Answer: 230

The 23rd multiple of ten means 23 tens.

tens	ones
23	

If we want to write the answer in place value notation we have to keep exchanging 10 tens for a hundred until there are less than 10 tens remaining. This gives us:

hundreds	tens	ones
	23	
1	13	
2	3	

which we read as 230.

Notice that we would get the same answer if we simply placed a 0 after 23.

- - *The Multiples of a Hundred*

These are the numbers we get if we count by hundreds. That is:

1 *hundred*, 2 *hundreds*, 3 *hundreds*, . . .

or

100, 200, 300, . . .

Example 21

What is the 23rd multiple of a hundred?

Answer: 2,300

The 23rd multiple of a hundred means 23 hundreds. Using the same approach as in Example 14, we have:

thousands	hundreds	tens	ones
	23		
1	13		
2	3		

which we write in place value notation as:

2,300

- - *Multiples of 10^n*

Recall that 10^n is the place value numeral that consists of a 1 followed by n zeroes (Module 1)

These are the numerals that consist of the sequence:

1 followed by n zeroes, 2 followed by n zeroes, 3 followed by n zeroes, . . .

Example 22

What is the 8th multiple of 10^6?

Answer: 8,000,000

Since 10^6 means a 1 followed by 6 zeroes, we have that $10^6 = 1{,}000{,}000$. Hence the 8th multiple fo 10^6 means eight 1,000,000's, or 8 million. In place value notation, 8 million is 8,000,000.

A good idea is simply to write in the correct number of 0's first, and place the commas properly.

In effect, we replaced the 1 in 1,000,000 by an 8.

Perhaps by now you've begun to notice a pattern.

A Pattern

Place value notation gives as an easy way to recognize multiples of 10^n. Namely:

- - *If the number ends in one 0 it is a multiple of ten.*
- - *If the number ends in two 0's it is a multiple of 100*
- - *If the number ends in n zeroes, it is a multiple of 10^n.*

That is, the ones-digit is 0.

That is, both the ones-digit and tens-digit are 0's

Notice that a number can be a multiple of more than one power of ten.

Example 23

Is 45,100 a multiple of:

(a) ten? (b) hundred? (c) thousand?

Answer: (a) yes (b) yes (c) no

(a) Since 45,100 ends in a 0, it is a

multiple of ten. In fact it is the 4,510th multiple of ten. To see this, simply cross out the last 0 and look at what number is left--namely 4,510.

Don't let the comma confuse you. Read 45,100 as 45100. Cross out the last 0 to get 4510~~0~~ or 4510. Then insert the comma in the proper place.

(b) Since 45,100 ends in two 0's it is a multiple of a hundred. If we cross out the last two 0's we see that 45,100 is the 451st multiple of a hundred.

This means that if you coun by hundreds; 45,100 is the 451st number you come to.

(c) Since 45,100 doesn't end in 3 zeroes it is not a multiple of a thousand. In fact the 45th multiple of a thousand is 45 thousand (45,000) which is less than 45,100; while the 46th multiple of a thousand is 46 thousand (46,000) which is greater than 45,100.

Example 24

Between what two consecutive multiples of a thousand is 45,100?

Answer: 45,000 and 46,000

This is a key point. Any number which is not a multiple of 10^n must be between two consecutive multiples of 10^n. In Example 16 we've already seen that 45,100 is between the 45th multiple of a thousand and the 46th multiple of a thousand.

45 and 46 are called consecutive multiples because there are no whole numbers between 45 and 46.

Now comes the idea of rounding off. In terms of thousands we know that 45,100 is between 45,000 and 46,000. The difference of 45,100 and 45,000 is only 100 while the difference between 46,000 and 45,100

is 900. This tells us that 45,100 is "closer" or "more nearly equal" to 45,000 than to 46,000.

If we replace 45,100 by 45,000 we say that we have rounded 45,100 off to the nearest (multiple of a) thousand.

We usually say "to the nearest thousand" rather than to the nearest multiple of a thousand"

Example 25

Round 482 off to the nearest hundred.

Answer: 500

The multiples of a hundred end in two 0's.

We have: 100, 200, 300, 400, 500, . . .

Clearly 482 is between 400 and 500.

Since 500 - 482 = 18 and 482 - 400 is 82, we see that 482 is closer to 500 than to 400.

In more formal language, 482 is greater than the 4th multiple of a hundred (400) but less than the 5th multiple of a hundred (500)

Note:

It would also be correct to say that 482 is between 200 and 700; but 200 and 700 are not consecutive multiples of a hundred.

That is, 4 and 5 are consecutive numbers but 2 and 7 aren't.

Sometimes we want a better approximation than to the nearest hundred.

Example 26

Round off 482 to the nearest ten.

Answer: 480

The multiples of ten end in one 0.

That is: 10, 20, 30, . . . Of course it's going to take a long time to get close to 482 this way, so we skip a few steps. We know that 482 is between 480 and 490 (the 48th and 49th multiples of ten). 482 - 480 = 2 but 490 - 482 = 8. So to the nearest ten, 482 is closer to 480 than to 490.

In more detail:

480
481
482 ← *We can see that 482 is closer to 480 than to 490.*
483
484
485
486
487
488
489
490

Example 27

Round off 52,876 to the nearest hundred.

Answer: 52,900

Multiples of a hundred end in at least two 0's. So we see that 52,876 is between 52,800 (the 528th multiple of a hundred) and 52,900 (the 529th multiple of a hundred).

A quick way to find 52,800 is to start with 52,876 and replace the last two digits by 0's to get 52,800.

But 52,876 is closer to 52,900 than to 52,800. That is:

52,900 - 52,876 = 24

while

52,876 - 52,800 = 76

Note:

The two differences, 24 and 76, add up to 100, checking with the fact that we're rounding off to the nearest hundred.

A shortcut is to look at the digit immediately to the right of the rounding off place. If this digit is less than 5 pick the lesser multiple; if it's greater than 5, pick the greater multiple.

Now let's get back to the problem that started this discussion.

Example 28

Round off each of the following two numbers to the nearest billion:

(a) 2,119,435,982

(b) 4,967,059,231

Answers (a) 2 billion
(b) 5 billion

Recall that a billion is 1,000,000,000. Hence multiples of a billion end in at least nine 0's.

(a) Replacing the last nine digits in 2,119,435,982 by 0's, we see that it is between 2,000,000,000 and 3,000,000,000.

That is: 2,119,435,982
↓↓↓ ↓↓↓ ↓↓↓
2,000,000,000

2,119,435,982 is closer to 2,000,000,000 than to 3,000,000,000 because the digit immediately to the right of the billions-place is 1, which is less than 5.

We could, of course, do this more directly, by finding the differences:
3,000,000,000 - 2,119,435,982
and
2,119,435,982 - 2,000,000,000

(b) By the same reasoning 4,967,059,231 is between 4,000,000,000 and 5,000,000,000. The digit immediately to the right of the billions-place is 9, so we are closer to 5 billion than to 4 billion.

Now for the punch line!

<u>Example 29</u>

By rounding off each number to
the nearest billion, estimate the
sum of 2,119,435,982 and 4,967,059,231.

Answer: The estimate is 7 billion; that is: 7,000,000,000

From the previous example we have that to the nearest billion, 2,119,435,982 is 2,000,000,000 and that 4,967,059,231 is 5,000,000,000.

Using $\doteq$ to stand for "approximately equal", we have that:

"approximately" is a vague term. In the context of this example, it means to the nearest billion.

$$2,119,435,982 \doteq 2,000,000,000$$
and
$$4,967,059,231 \doteq 5,000,000,000$$

So our estimate of the sum in this case is 2,000,000,000 + 5,000,000,000 or 7,000,000,000.

In essence we're replacing a cumbersome arithmetic problem by the simpler problem of adding 2 and 5; or more precisely: 2 billion + 5 billion = 7 billion

Additional drill is left to the Self-Test.

With the advent of the electronic calculator there has been a general feeling that "old-fashioned" arithmetic is passe. The feeling essentially says:

> "Why bother to learn how do arithmetic, when the calculator does it faster and better? Moreover, why bother to learn how to round off when it isn't that much work to get the exact answer using the calculator?"

A healthier attitude might be that because of the calculator we should spend less time on drill and more on reading comprehension and problem solving. This will be discussed in more detail in Module 9, at whic time you may start to use a calculator in this course.

While this argument has some merit, it overlooks several possibilities--

(1) What happens if the numbers have more digits than the capacity of your calculator? For example, if your calculator can only display 12 digits, how would you use it to add 2,345,678,209,567,349 and 15,679,234,609?

But by rounding off we know that the answer is approximately 2 quadrillion plus 16 billion.

(2) Even when there are few enough digits for your calculator to handle, you might make an error in entering the digits. Just as people can make typographical errors, they can make calculator errors. By knowing how to make estimates, we get an idea of the correct answer before we start the problem.

This will not prevent "small" errors but it will help us guard against preposterous errors.

Perhaps of even more importance, calculator errors don't "jump out at you" like typographical errors. For example, if you transpose the a and h in "that" to get "taht" you can probably figure out the error by the context of the word. On the other hand if you transposed the 9 and 4 in "4,967,059,231" to get "9,467,059,231" you might not even suspect an error unless you have some number sense.

(3) By "reasoning" the answer, you help develop a number sense that is crucial in solving real-life problems. In fact, in the real-world, it is usually a <u>verbal</u> problem that first has to be interpreted numerically before we can even think of using a calculator. In effect, using the calculator when you don't understand the problem can result in your finding the answer to the <u>wrong</u> problem.

And so we come to the end of this Module on a note that is a theme for this course. We never have to choose between the "old" and the "new". Rather we take the best of each and combine them. In essence, the progress of humankind is to assimilate new material with the successful parts of the old material. In terms of this Module, we see the calculator and other modern devices as supplements to "old" mathematics.

In this spirit, we shall work on developing our number sense for the next several modules. During this time you should not use the calculator to do the examples and Test Problems. But once sufficient time has been spent on helping you develop a number sense, you will be encouraged to use a calculator--at which time our attention shall switch to the topic of how to use arithmetic in order to solve real-world problems.

After you've done the problems, however, you may use the calculator as a way to check your answers.